Stabilities of generalized entropies

2004 J. Phys. A: Math. Gen. 37 10513

(http://iopscience.iop.org/0305-4470/37/44/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10513–10519 PII: S0305-4470(04)78846-0

Stabilities of generalized entropies

Sumiyoshi Abe[1], **G Kaniadakis**[2] **and A M Scarfone**[2]

[1] Institute of Physics, University of Tsukuba, Ibaraki 305-8571, Japan
[2] Dipartimento di Fìsica and Istituto Nazionale di Fìsica della Materia (INFM), Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

Received 5 April 2004, in final form 16 September 2004
Published 20 October 2004
Online at stacks.iop.org/JPhysA/37/10513
doi:10.1088/0305-4470/37/44/004

Abstract
The generalized entropic measure, which is maximized by a given arbitrary distribution under the constraints on normalization of the distribution and the finite ordinary expectation value of a physical random quantity, is considered. To examine if it can be of physical relevance, its experimental robustness is discussed. In particular, Lesche's criterion is analysed, which states that an entropic measure is stable if its change under an arbitrary weak deformation of the distribution (representing fluctuations of experimental data) remains small. It is essential to note the difference between this criterion and thermodynamic stability. A general condition, under which the generalized entropy becomes stable, is derived. Examples known in the literature, including the entropy for the stretched-exponential distribution, the quantum-group entropy and the κ-entropy are discussed.

PACS numbers: 65.40.Gr, 02.50.−r, 05.20.−y, 05.90.+m

1. Introduction

There is great diversity in statistical distributions observed in nature. This is apparently a challenge for statistical mechanics. In view of traditional statistical mechanics based on the Boltzmann–Gibbs–Shannon entropy, a significant number of distributions observed in complex systems are actually anomalous. A few examples are granular materials, glasses, self-gravitating systems, biological systems and seismicity. An important point here is that these anomalous distributions can persist for very long periods of time which are much longer than typical time scales of underlying microscopic dynamics. This fact naturally leads to the question if there could be a framework for understanding such diverse statistical phenomena in a unified manner. In this respect, the principle of maximum entropy pioneered by Gibbs and Jaynes may be thought of as one such [1]. Then, if one would wish to describe such anomalous distributions based on the principle of maximum entropy, there might be at least two ways to address this. One is to modify the form of the constraints, and the other is to generalize the

Boltzmann–Gibbs–Shannon entropy. The latter is the standpoint which we shall take in the present work.

In a recent paper [2], one of the present authors has proposed an entropy-generating algorithm. A very general class of entropic measures, which are maximized by given distributions under the appropriate constraints on normalization and the ordinary expectation value of a physical random variable such as the energy, was constructed. However, in spite of the mathematical consistency of the discussion, it is still to be clarified if the entire class of such generalized entropies may be good measures both physically and mathematically.

In this paper, we examine the concept of stability proposed by Lesche in [3] (see also [4]), which should be satisfied by any physical entropic measure. We shall derive a general condition, under which the generalized entropy can satisfy the Lesche stability property (that is different from thermodynamic stability). The stability properties of some examples, including the entropy for the stretched-exponential distributions [5], the quantum-group entropy [6] and the κ-entropy [7–10] are also discussed.

2. Generalized entropy

In [2], an algorithm has been presented for generating a generalized entropy which is maximized by a given arbitrary distribution under the constraints on normalization of the distribution and the ordinary expectation value of a physical quantity, $\{Q_i\}_{i=1,2,\ldots,W}$, of interest (e.g., the system energy), where W is the number of microscopically accessible states.

Given a normalized distribution $\{p_i = f(\alpha + \beta Q_i)\}_{i=1,2,\ldots,W}$, the corresponding generalized entropy maximized by it is constructed as follows:

$$S[p] = \int_{t_{\min}}^{t_{\max}} \mathrm{d}t\,(1 - A[p;t]) + c. \tag{1}$$

Here, α and β are the Lagrange multipliers associated with the constraints on normalization and the ordinary expectation value of $\{Q_i\}_{i=1,2,\ldots,W}$, respectively. $A[p;t)$ is a quantity given by

$$A[p;t) = \sum_{i=1}^{W} (p_i - f(t))_+ \tag{2}$$

with the notation

$$(x)_+ = \max\{0, x\}. \tag{3}$$

c is the constant which should be determined in such a way that $S[p]$ vanishes for the completely ordered state, $p_i = p_i^{(0)} = \delta_{ij}$ $(1 \leqslant j \leqslant W)$. $f(t)$ is a function that determines the form of the distribution, p_i. For the sake of simplicity, this function is assumed to be a monotonically decreasing function with the range (0, 1) and to satisfy the condition

$$\int_{t_{\min}}^{t_{\max}} \mathrm{d}t\, f(t) < \infty, \tag{4}$$

where $(t_{\min}, t_{\max})$ is the domain of $f(t)$, i.e., $f(t) \to 1(0)$ as $t \to t_{\min}(t_{\max})$.

It can be seen that equation (1) is written in the following form:

$$S[p] = \sum_{i=1}^{W} \left[p_i f^{-1}(p_i) - \int_{f^{-1}(0)}^{f^{-1}(p_i)} \mathrm{d}t\, f(t) \right] + \int_{f^{-1}(0)}^{f^{-1}(1)} \mathrm{d}t\, f(t) - f^{-1}(1), \tag{5}$$

where f^{-1} is the inverse function of f. Moreover, if f^{-1} is piecewise differentiable, as assumed here and hereafter, then $S[p]$ can be further rewritten as follows[3]:

$$S[p] = \sum_{i=1}^{W} \int_0^{p_i} \mathrm{d}t\, f^{-1}(t) - \int_0^1 \mathrm{d}t\, f^{-1}(t). \tag{6}$$

With this form, it is now evident that the stationarity condition on the functional under the constraints on the normalization condition $\sum_{i=1}^{W} p_i = 1$ and on the ordinary expectation value $\langle Q\rangle = \sum_{i=1}^{W} p_i Q_i$: $\delta\big(S[p] - \alpha \sum_{i=1}^{W} p_i - \beta \sum_{i=1}^{W} p_i Q_i\big) = 0$, which in fact yields the maximal distribution $\{p_i = f(\alpha + \beta Q_i)\}_{i=1,2,\ldots,W}$, where α and β are the Lagrange multipliers associated with the two constraints.

The construction in equation (1) with equation (2) is a general mathematical procedure for generating a concave functional of $\{p_i\}_{i=1,2,\ldots,W}$. In addition, as shown in [2], $S[p]$ satisfies the H-theorem for the master equation combined with the principle of microscopic reversibility.

3. Stability criterion

It is not expected that whole class of generalized entropic measures expressed in the form in equation (1) or equation (6) are physically relevant, even though they are concave and satisfy the H-theorem. In order for an entropic measure to be experimentally robust, it is necessary for the measure to satisfy the stability condition proposed in [3]. This concept is stated as follows. Usually, what is experimentally measured is not directly a statistical entropy, Σ, itself but a distribution of the values of a physical quantity. Repeating the same experiment and observing the same physical quantity, an experimentalist will obtain a distribution which may be slightly different than that observed previously. If Σ is of physical relevance, then at least its value should not change drastically for two slightly different distributions, $\{p_i\}_{i=1,2,\ldots,W}$ and $\{p'_i\}_{i=1,2,\ldots,W}$. Mathematically, this implies

$$(\forall\varepsilon > 0)(\exists\,\delta > 0)\left(\|p - p'\|_1 < \delta \Rightarrow \left|\frac{\Sigma[p] - \Sigma[p']}{\Sigma_{\max}}\right| < \varepsilon\right) \tag{7}$$

for any value of W, where $\|A\|_1 = \sum_{i=1}^{W} |A_i|$ and $\Sigma_{\max}$ is the maximum value of Σ. To examine this condition for the quantity in equation (1), we analyse the following inequality:

$$|S[p] - S[p']| \leqslant \int_{t_{\min}}^{\tau} \mathrm{d}t |A[p;t) - A[p';t)| + \int_{\tau}^{t_{\max}} \mathrm{d}t |A[p;t) - A[p';t)|, \tag{8}$$

where τ satisfies

$$t_{\min} < f^{-1}(1/W) \leqslant \tau < t_{\max}. \tag{9}$$

To evaluate the right-hand side of equation (8), we note the following properties:

$$|A[p;t) - A[p';t)| \leqslant \|p - p'\|_1, \tag{10}$$

$$|A[p;t) - A[p';t)| \leqslant Wf(t) \qquad (t \geqslant f^{-1}(1/W)), \tag{11}$$

which, respectively, follow from the facts that $|(x)_+ - (y)_+| \leqslant |x - y|$ and $1 - Wf(t) \leqslant A[p;t) < 1$ arising from the relations: $(1 - Wf(t))_+ = \big(\sum_{i=1}^{W} (p_i - f(t))\big)_+ \leqslant \sum_{i=1}^{W} (p_i - f(t))_+ < 1$. Using equations (10) and (11) in equation (8), we find

$$|S[p] - S[p']| \leqslant G(\tau), \tag{12}$$

[3] In [2], it was mentioned that the functional in equation (6) is not concave, in general. However, under the differentiability condition, the quantity in equation (6) in fact turns out to be a concave functional, as shown in [2].

$$G(\tau) = (\tau - t_{\min})\|p - p'\|_1 + W \int_{\tau}^{t_{\max}} \mathrm{d}t\, f(t). \tag{13}$$

Equation (12) holds for any value of τ satisfying equation (9). Let us take $\tau = \tau_0$ which makes $G(\tau)$ minimum:

$$\tau_0 = f^{-1}(\|p - p'\|_1/W). \tag{14}$$

Therefore, we have

$$\begin{aligned} |S[p] - S[p']| &\leqslant \{f^{-1}(\|p - p'\|_1/W) - t_{\min}\}\|p - p'\|_1 + W \int_{f^{-1}(\|p-p'\|_1/W)}^{t_{\max}} \mathrm{d}t\, f(t) \\ &= W t_{\max} f(t_{\max}) - t_{\min}\|p - p'\|_1 - W \int_{\|p-p'\|_1/W}^{f(t_{\max})} \mathrm{d}t\, f^{-1}(t). \end{aligned} \tag{15}$$

Noting that equation (6) takes the following maximum value for the equiprobability:

$$S_{\max} = W \int_0^{1/W} \mathrm{d}t\, f^{-1}(t) - \int_0^1 \mathrm{d}t\, f^{-1}(t), \tag{16}$$

we have

$$\left| \frac{S[p] - S[p']}{S_{\max}} \right| \leqslant B(\|p - p'\|_1, W), \tag{17}$$

in which we have introduced

$$B(\|p - p'\|_1, W) \equiv \frac{\int_0^{\|p-p'\|_1/W} \mathrm{d}t\, f^{-1}(t) - (t_{\min}/W)\|p - p'\|_1}{\int_0^{1/W} \mathrm{d}t\, f^{-1}(t) - (1/W)\int_0^1 \mathrm{d}t\, f^{-1}(t)}, \tag{18}$$

where we have used the fact that $tf(t)$ tends to vanish in the limit $t \to t_{\max}$, due to equation (4) as well as the property, $f(t) \to 0$ $(t \to t_{\max})$. Therefore, we conclude that the generalized entropy is stable in the thermodynamic limit, $W \to \infty$, if

$$\lim_{\|p-p'\|_1 \to +0} \lim_{W \to \infty} B(\|p - p'\|_1, W) = 0. \tag{19}$$

This is our main result. Note that this order of taking the limits is essential for the Lesche stability criterion.

Note that $B(\|p - p'\|_1, W)$ in equation (18) is an indeterminate form in the limit $W \to \infty$. A particular case when application of L'Hôpital's rule once to this limit is sufficient, then we have

$$\left| \frac{S[p] - S[p']}{S_{\max}} \right| \leqslant C\|p - p'\|_1, \tag{20}$$

$$C = \frac{f^{-1}(+0) - f^{-1}(1\text{–}0)}{f^{-1}(+0) - \int_0^1 \mathrm{d}t\, f^{-1}(t)}. \tag{21}$$

So, in this case, taking δ as $\delta = \varepsilon/C$, we see that the generalized entropy satisfies the stability condition in equation (7).

Closing this section, we note that in [11] the continuity and stability properties of a class of generalized entropies are discussed by employing an approach different from the present one.

4. Examples

In this section, we discuss some examples of stable generalized entropies known in the literature.

4.1. Entropy for stretched-exponential distribution

The stretched-exponential distribution, $p_i \sim \exp(-|\alpha + \beta Q_i|^\gamma)$ ($\gamma \in (0, 1)$), is known to play important roles in physical problems such as turbulence [12] and fragmentation [13]. Here, $f(t)$ is taken to be

$$f(t) = \exp(-t^\gamma), \tag{22}$$

where $t \in (t_{\min}, t_{\max}) = (0, \infty)$. Substitution of this function into equation (6) gives rise to the following generalized entropy [2, 5]:

$$S_{\mathrm{SE}}[p] = \sum_{i=1}^{W} \Gamma(1 + 1/\gamma, -\ln p_i) - \Gamma(1 + 1/\gamma), \tag{23}$$

where $\Gamma(u, x)$ is the incomplete gamma function of the second kind, $\Gamma(u, x) = \int_x^\infty \mathrm{d}t\, t^{u-1}\, \mathrm{e}^{-t}$, and $\Gamma(u) = \Gamma(u, 0)$ is the ordinary gamma function. Since $f^{-1}(t) = (-\ln t)^{1/\gamma}$ with $t \in (0, 1)$, C in equation (21) is calculated to be

$$C = \lim_{t \to +0} \frac{f^{-1}(t)}{f^{-1}(t) - \Gamma(1 + 1/\gamma)} = 1. \tag{24}$$

Therefore, taking $\delta = \varepsilon$, the entropy for the stretched-exponential distribution is seen to satisfy the Lesche stability condition.

In the particular case when $\gamma \to 1$–0, $S_{\mathrm{SE}}[p]$ converges to the Boltzmann–Gibbs–Shannon entropy, $S_{\mathrm{BGS}}[p] = -\sum_{i=1}^{W} p_i \ln p_i$, as it should do. In turn, as a byproduct, stability of the Boltzmann–Gibbs–Shannon entropy shown in [3] is ascertained.

4.2. Quantum-group entropy

The quantum-group entropy is given by

$$S_{\mathrm{QG}}[p] = -\sum_{i=1}^{W} \frac{(p_i)^q - (p_i)^{q^{-1}}}{q - q^{-1}}. \tag{25}$$

This quantity has been introduced in [6] and has been applied there to generalized statistical-mechanical study of q-deformed oscillators. A basic idea is to incorporate the nonadditive feature of the energies of the systems having the quantum-group structures with generalized statistics. In equation (25), q is assumed to be positive. Since $S_{\mathrm{QG}}[p]$ is symmetric under interchange, $q \leftrightarrow q^{-1}$, the range of q can be reduced to (0, 1). This quantity also converges to the Boltzmann–Gibbs–Shannon entropy in the limit $q \to 1 - 0$.

The function, $f(t)$, defined on $(t_{\min}, t_{\max}) = (-1, \infty)$ associated with the quantum-group entropy is implicitly given as the inverse function of

$$f^{-1}(t) = -\frac{q t^{q-1} - q^{-1} t^{q^{-1}-1}}{q - q^{-1}}. \tag{26}$$

For this function, C in equation (21) is still an indeterminate form. Accordingly, $B(\|p - p'\|_1, W)$ in equation (18) is evaluated directly as follows:

$$\begin{aligned} B(\|p - p'\|_1, W) &= \frac{\int_0^{\|p-p'\|_1/W} \mathrm{d}t\, f^{-1}(t) + (1/W)\|p - p'\|_1}{\int_0^{1/W} \mathrm{d}t\, f^{-1}(t)} \\ &= \frac{(\|p - p'\|_1/W)^q - (\|p - p'\|_1/W)^{q^{-1}} - (q - q^{-1})(1/W)\|p - p'\|_1}{W^{-q} - W^{-q^{-1}}} \\ &\to (\|p - p'\|_1)^q \qquad (W \to \infty). \end{aligned} \tag{27}$$

Therefore, taking $\delta = \varepsilon^{1/q}$, we see that the quantum-group entropy is stable.

4.3. κ-entropy

The κ-entropy has been introduced in [7] and has been applied to systems described by statistical distributions having a power-law asymptotic behaviour. An important worked physical example is the energy distributions of the fluxes of cosmic rays [7] (see also [14]). In [7], the following one-parameter generalizations of the ordinary exponential and logarithmic functions have been proposed:

$$\exp_{\{\kappa\}}(t) = \left(\sqrt{1+\kappa^2 t^2} + \kappa t\right)^{1/\kappa}, \tag{28}$$

$$\ln_{\{\kappa\}}(t) = \frac{t^{\kappa} - t^{-\kappa}}{2\kappa}, \tag{29}$$

from which the ordinary exponential and logarithmic functions are, respectively, reproduced in the limit $\kappa \to +0$. κ should be in the range $(-1, 1)$. On the other hand, $\exp_{\{\kappa\}}(t)$ is defined for $t \in (-\infty, \infty)$ and $\ln_{\{\kappa\}}(t)$ for $t \in (0, \infty)$. Both of these functions are symmetric under interchange $\kappa \leftrightarrow -\kappa$. (For $\exp_{\{\kappa\}}(t)$, this interchange is combined with the reversal of t.) Therefore, the range of κ can be taken to be $(0, 1)$.

It is a simple task to verify that if we choose

$$f(t) = \exp_{\{\kappa\}}(-t), \tag{30}$$

$$f^{-1}(t) = -\ln_{\{\kappa\}}(t) \qquad \text{for} \quad t \in (t_{\min}, t_{\max}) = (0, \infty) \tag{31}$$

then we obtain the κ-entropy given by

$$S_{\kappa}[p] = -\sum_{i=1}^{W} \{c_{-\kappa}[(p_i)^{1-\kappa} - p_i] + c_{\kappa}[(p_i)^{1+\kappa} - p_i]\}, \tag{32}$$

$$c_{\kappa} = \frac{1}{2\kappa(1+\kappa)}. \tag{33}$$

Substituting equation (31) into equation (18), we find

$$B(\|p - p'\|_1, W) \to (\|p - p'\|_1)^{1-\kappa} \qquad (W \to \infty). \tag{34}$$

Therefore, setting $\delta = \varepsilon^{1/(1-\kappa)}$, we conclude that the κ-entropy is stable [10]. It is worth mentioning that the κ-entropy becomes reduced to the Boltzmann–Gibbs–Shannon entropy in the limit $\kappa \to +0$.

5. Concluding remarks

We have discussed the generalized entropy maximized by a given arbitrary distribution under the constraints on normalization of the distribution and the ordinary expectation value of a physical random quantity. We have examined its Lesche stability property and have derived a general condition, under which the generalized entropy becomes stable. We have also discussed some examples of entropic measures known in the literature and have established their stabilities.

In recent years, much attention has been paid to the Tsallis entropy [15], which has been employed for generalizing Boltzmann–Gibbs statistical mechanics for nonextensive systems [16–18]. It has been shown in [4] that the Tsallis entropy is stable.

There are also unstable entropies. Examples are the Rényi entropy [19] and the so-called normalized Tsallis entropy [20, 21], whose instabilities have been shown in [3] and

[4], respectively. Quite interestingly, these quantities cannot be expressed in the form in equation (1) and are not concave if their entropic indices are larger than unity.

Finally, we point out that mathematically the Lesche stability property is equivalent to uniform continuity of the functional under consideration. The problem of continuity itself has recently been studied in [22, 23], where the Boltzmann–Gibbs–Shannon entropy is shown not to be continuous for infinite microscopic states.

Acknowledgment

SA was supported in part by the grant-in-aid for Scientific Research of Japan Society for the Promotion of Science.

References

[1] Rosenkrantz R D (ed) 1989 *E. T. Jaynes: Papers on Probability, Statistics and Statistical Physics* (Dortrecht: Kluwer)
[2] Abe S 2003 *J. Phys. A: Math. Gen.* **36** 8733
[3] Lesche B 1982 *J. Stat. Phys.* **27** 419
[4] Abe S 2002 *Phys. Rev.* E **66** 046134
[5] Anteneodo C and Plastino A R 1999 *J. Phys. A: Math. Gen.* **32** 1089
[6] Abe S 1997 *Phys. Lett.* A **224** 326
Abe S 1998 *Phys. Lett.* A **244** 229
[7] Kaniadakis G 2001 *Physica* A **296** 405
Kaniadakis G 2002 *Phys. Rev.* E **66** 056125
[8] Kaniadakis G and Scarfone A M 2001 *Physica* A **305** 69
[9] Naudts J 2002 *Physica* A **316** 323
[10] Kaniadakis G and Scarfone A M 2003 *Physica* A **340** 102 (*Preprint* cond-mat/0310728)
[11] Naudts J 2004 *Rev. Math. Phys.* **16** 809 (*Preprint* math-ph/0208038)
[12] Bohr T, Jensen M H, Paladin G and Vulpiani A 1998 *Dynamical Systems Approach to Turbulence* (Cambridge: Cambridge University Press)
[13] Sornette D 2000 *Critical Phenomena in Natural Sciences* (Berlin: Springer)
[14] Tsallis C, Anjos J C and Borges E P 2003 *Phys. Lett.* A **310** 372
[15] Tsallis C 1988 *J. Stat. Phys.* **52** 479
[16] Abe S and Okamoto Y (ed) 2001 *Nonextensive Statistical Mechanics and Its Applications* (Springer: Heidelberg)
[17] Kaniadakis G, Lissia M and Rapisarda A (ed) 2002 *Physica* A **305** (special issue)
[18] Gell-Mann M and Tsallis C (ed) 2004 *Nonextensive Entropy—Interdisciplinary Applications* (Oxford: Oxford University Press)
[19] Rényi A 1970 *Probability Theory* (Amsterdam: North Holland)
[20] Landsberg P T and Vedral V 1998 *Phys. Lett.* A **247** 211
[21] Rajagopal A K and Abe S 1999 *Phys. Rev. Lett.* **83** 1711
[22] Harremoës P and Topsøe F 2001 *Entropy* **3** 191
[23] Harremoës P 2004 *János Bolyai Mathematical Society* at press